AN EXPOSITION OF A PROOF OF GABRIEL'S THEOREM

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Abstract. This paper is a short exposition containing a proof of Gabriel's theorem, a foundational theorem in quiver representation theory. In 1972, Bernstein, Gelfand and Ponomarev published a paper containing a proof of Gabriel's Theorem, among other things. We give background and motivation for the theorem and prove it in a way adapted from their paper, but with additional exposition at key steps and parts reordered for clarity.

Contents

1. Introduction

A **quiver** (Γ, Λ) is a directed graph where edges from a vertex to itself and multiple edges between the same pair of vertices are permitted. Formally, it is a set of vertices Γ_0, a set of unoriented edges Γ_1 (together forming a graph Γ), and an **orientation** Λ that assigns to each edge l in Γ_1 a starting vertex $\alpha(l)$ and an ending vertex $\beta(l)$. Fixing a field k, a **representation** (V, f) of a quiver is a collection of vector spaces V_α over k for every α in Γ_0 and a collection of linear maps $f_l : V_{\alpha(l)} \to V_{\beta(l)}$ for every l in Γ_1. For any two representations (V, f) and (U, g) of (Γ, Λ), we define their direct sum to be the representation (W, h), where we have $W_\alpha = V_\alpha \oplus U_\alpha$ and $h_l = f_l \oplus g_l$. We call a nonzero representation **indecomposable** if it cannot be written as the direct sum of two representations. The **dimension vector** of a representation (V, f) is the tuple x of natural numbers indexed by the vertices Γ_0, where the component x_α is equal to $\dim V_\alpha$. Where it will not lead to

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confusion, we often write V for a representation instead of (V, f). If the dimension vector of a representation V is x, we write $\dim V = x$.

A **morphism of representations** φ between two representations (V, f) and (W, g) of (Γ, Λ) is a collection of linear maps φ_α with a map for every α in Γ_0 such that for every l in Γ_1 the diagram

$$\begin{tikzcd}
V_{\alpha(l)} \arrow[r, "f_l"] \arrow[d, "\varphi_{\alpha(l)}"'] & V_{\beta(l)} \arrow[d, "\varphi_{\beta(l)}"] \\
W_{\alpha(l)} \arrow[r, "g_l"'] & W_{\beta(l)}
\end{tikzcd}$$

commutes. If all the φ_α are isomorphisms, we say that φ is an **isomorphism of representations**. For every quiver (Γ, Λ), the representations of (Γ, Λ) together with the morphisms between those representations define a category $\mathscr{L}(\Gamma, \Lambda)$.

So, why study quivers? It turns out that quivers have useful applications to the study of the representation theory of finite-dimensional algebras. Associated with every quiver $Q := (\Gamma, \Lambda)$ and choice of field k is a k-algebra kQ called its **path algebra**, which we will not define here. There is a bijection between representations of Q and left kQ-modules [2]. Two algebras A, B are said to be **Morita equivalent** if the category of left A-modules and the category of left B-modules are equivalent [4]. It turns out that every algebra is Morita equivalent to some path algebra, so many questions about modules over general algebras can be reformulated into questions about representations of quivers. This is often a useful strategy due to the surprising fact that all modules of path algebras have finite and easy-to-compute projective resolutions called Ringel resolutions [2]. Additional discussion of the notion of Morita equivalence can be found in [4], and a precise definition of the Ringel resolution can be found in [2].

It is a central theorem of representation theory that, under certain nice conditions, every representation can be written uniquely as a sum of indecomposable representations. The formal statement and proof of this theorem (the Krull-Schmidt theorem) is given as Theorem 19.21 of [3].

We call a quiver with only finitely many isomorphism classes of indecomposable representations a **finite type** quiver.

Given that quivers make working with general algebras easier, and that finite type quivers are the easiest quivers to work with, it is a natural next step to classify the finite type quivers.

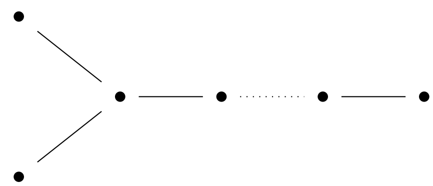
2. Gabriel's Theorem

Theorem 2.1 (Gabriel's Theorem)**.** *A quiver* (Γ, Λ) *is of finite type if and only if* Γ *is one of the following graphs:*

- A_n*:*

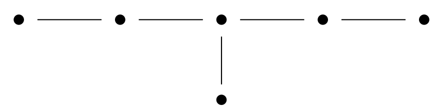
(n total vertices)

- *D_n:*

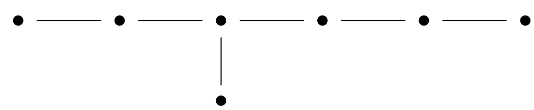


(n total vertices)

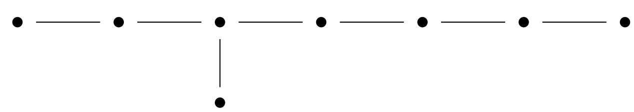
- *E_6:*



- *E_7:*



- *E_8:*



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These graphs are called **simply laced Dynkin diagrams***.*

Remark 2.2. In particular, the orientation of a quiver does not affect whether it is finite type.

Remark 2.3. Moreover, at the end of the proof, we will be able to easily determine whether a given representation of a quiver (Γ, Λ) is indecomposable, so long as Γ is a simply laced Dynkin diagram.

This paper will follow the proof given in [1], with additional exposition given along the way. Although the theorem is true for general fields k, we only give the proof in the case where $k = \mathbb{R}$ or $k = \mathbb{C}$.

3. Proof that all finite type quivers are oriented simply laced Dynkin diagrams

First, we give some definitions.

Definition 3.1. Let (Γ, Λ) be a quiver. We denote by $\mathbb{Q}^{\Gamma_0}$ the vector space over $\mathbb{Q}$ with dimension $|\Gamma_0|$ and with basis indexed by Γ_0. This space contains the dimension vector of each representation of (Γ, Λ).

Definition 3.2. We denote by $\overline{\beta}$ the basis vector indexed by β in $\mathbb{Q}^{\Gamma_0}$.

Definition 3.3. We denote by B the quadratic form on the space $\mathbb{Q}^{\Gamma_0}$ defined by

$$B(x) = \sum_{\alpha \in \Gamma_0} x_\alpha^2 - \sum_{l \in \Gamma_1} x_{\alpha(l)} x_{\beta(l)},$$

abusing notation slightly since Γ is not oriented.

Now we can restate and prove the first direction of Gabriel's Theorem.

Theorem 3.4. *If* (Γ, Λ) *is a finite type quiver, then* Γ *is a simply laced Dynkin diagram.*

Proof. Let $Q := (\Gamma, \Lambda)$ be such a quiver. We first show that the form B must be positive definite. Fix some dimension vector $x \in \mathbb{N}^{\Gamma_0}$. We denote by A the set of representations of Q with dimension vector x.

Fix a basis in the space k^{x_α} for every vertex $\alpha \in \Gamma_0$. Let φ be an isomorphism between two representations in A. Then φ is completely described by the collection of matrices associated with its components, the φ_α's. The set G of such collections of matrices is the group

$$G := \prod_{\alpha \in \Gamma_0} GL_{x_\alpha}(k).$$

This group acts on A by change of basis, where we have (for $\phi \in G$ and $(V, f) \in A$)

$$(\phi.V)_l = \phi^{-1}_{\beta(l)} \circ f_l \circ \phi_{\alpha(l)}.$$

By definition, two representations in A are isomorphic if and only if they are in the same G-orbit. Thus, by our assumption that (Γ, Λ) is finite type, there are only finitely many G-orbits. The data of a representation of Q with fixed dimension vector x, fixing a basis for each V_α, is just a collection of linear maps. The matrices associated with these maps have dimensions described by the components of x; in particular, for every edge l, we have an $x_{\alpha(l)}$-by-$x_{\beta(l)}$ matrix. Since $k \in \{\mathbb{R}, \mathbb{C}\}$, the set A has a manifold structure such that the G-action is smooth, so each G-orbit must have the same dimension as A. This is the only step where we use the fact that $k \in \{\mathbb{R}, \mathbb{C}\}$; a sketch of the proof in the general case is given in [1]. Thus we have

$$\dim G \geq \dim A.$$

Additionally, the group G has a one-dimensional subgroup consisting of scalar matrices that acts identically on A, so we can revise the inequality to

$$\dim G - 1 \geq \dim A.$$

Computing the dimensions of G and A, we get

$$\sum_{\alpha \in \Gamma_0} x_\alpha^2 - 1 \geq \sum_{l \in \Gamma_1} x_{\alpha(l)} x_{\beta(l)},$$

which implies

$$B(x) > 0.$$

This argument works for every $x \in \mathbb{N}^{\Gamma_0}$. Moreover, we can compute that for any $x \in \mathbb{Q}^{\Gamma_0}$,

$$B(x) \geq B(|x|).$$

Therefore, if Q is finite type, then B is positive definite. It remains to show that B is positive definite only if Γ is a simply laced Dynkin diagram. This fact (together with its converse) is proven as Proposition 2.1 of [1].

□

4. Proof that all oriented simply laced Dynkin diagrams are finite type quivers

First, we give some definitions. Let (Γ, Λ) be a quiver.

Definition 4.1. We denote by Γ^x either the set of edges in Γ adjacent to a vertex x, or the set of vertices adjacent to x, depending on context.

Definition 4.2. A **sink** (resp. **source**) is a vertex x of a quiver such that for every edge $l \in \Gamma^x$, we have $x = \beta(l)$ (resp. $x = \alpha(l)$).

Definition 4.3. For a vertex $\alpha \in \Gamma_0$, the orientation $\sigma_\alpha \Lambda$ of Γ is the orientation that differs from Λ just at the edges adjacent to α.

Definition 4.4. A **sink sequence** is a sequence of vertices $\alpha_1, ..., \alpha_k$ in (Γ, Λ) such that the vertex α_1 is a sink with respect to Λ, the vertex α_2 is a sink with respect to $\sigma_{\alpha_1}\Lambda$, the vertex α_3 is a sink with respect to $\sigma_{\alpha_2}\sigma_{\alpha_1}\Lambda$, and so on. A **source sequence** is defined similarly.

Definition 4.5. Let (Γ, Λ) be an oriented graph without oriented cycles. A **sink ordering** is an ordering of all the vertices of Γ such that $\alpha(l)$ has a greater index than $\beta(l)$ for every edge $l \in \Gamma_1$. Such an ordering defines a sink sequence.

Definition 4.6. A vector $x \in \mathbb{Q}^{\Gamma_0}$ is called **integral** if its components are all integers, and **positive** if its components are all nonnegative and $x \neq 0$.

Definition 4.7. For each $\beta \in \Gamma_0$, we denote by σ_β the linear transformation from $\mathbb{Q}^{\Gamma_0}$ to $\mathbb{Q}^{\Gamma_0}$ defined by

$$(\sigma_\beta x)_\gamma = x_\gamma, \gamma \neq \beta$$

and

$$(\sigma_\beta x)_\beta = -x_\beta + \sum_{l \in \Gamma^\beta} x_{\gamma(l)},$$

where $\gamma(l)$ denotes the vertex adjacent to l other than β.

Definition 4.8. We denote by $\mathscr{W}$ the group of transformations of $\mathbb{Q}^{\Gamma_0}$ generated by the set $\{\sigma_\beta | \beta \in \Gamma_0\}$. These form a group because every σ_β has $\sigma_\beta^2 = 1$.

Definition 4.9. An element of $\mathbb{Q}^{\Gamma_0}$ of the form $\sigma_{\beta_1}...\sigma_{\beta_n}\overline{\alpha}$ is called a **root**. Each basis vector $\overline{\alpha}$ is called a **simple root**.

Definition 4.10. The **simple representation** L_γ for a vertex $\gamma \in \Gamma_0$ is the representation with dimension vector $\overline{\gamma}$. Put differently, L_γ is the representation (V, f) with $f_l = 0$ for all l, with $V_\beta = 0$ for $\beta \neq \gamma$, and with V_γ equal to the one-dimensional vector space k.

In this section, our goal is to prove the second direction of Gabriel's Theorem, which we restate next.

Theorem 4.11. *Let (Γ, Λ) be a quiver where Γ is a simply laced Dynkin diagram. Then (Γ, Λ) is a finite type quiver. Moreover, the map that sends a representation to its dimension vector is a bijection between the set of isomorphism classes of indecomposable representations of (Γ, Λ) and the set of positive roots in $\mathbb{Q}^{\Gamma_0}$.*

We defer the proof to the end of the section.

Proposition 4.12. *For Γ a simply laced Dynkin diagram, the set of positive roots in $\mathbb{Q}^{\Gamma_0}$ is finite.*

Proof. Since all the σ_β maps have integer coefficients with respect to our chosen standard basis, the group $\mathscr{W}$ preserves the integer lattice $\mathbb{Z}^{\Gamma_0} \subset \mathbb{Q}^{\Gamma_0}$. We also have that $\mathscr{W}$ preserves B, since we have

$$\begin{aligned} B(\sigma_\gamma x) = B(x) - (B(x) - B(\sigma_\gamma x)) &= \left(\sum_{\alpha\in\Gamma_0} x_\alpha^2 - 2\sum_{\beta\in\Gamma^\gamma} x_\gamma x_\beta + \left(\sum_{\beta\in\Gamma^\gamma} x_\beta\right)^2 \right) \\ &\quad - \left(\sum_{l\in\Gamma_1} x_{\alpha(l)} x_{\beta(l)} - 2\sum_{\beta\in\Gamma^\gamma} x_\gamma x_\beta + \sum_{\beta\in\Gamma^\gamma}\left(x_\beta \sum_{\beta\in\Gamma^\gamma} x_\beta\right)\right) \\ &= \left(\sum_{\alpha\in\Gamma_0} x_\alpha^2 - 2\sum_{\beta\in\Gamma^\gamma} x_\gamma x_\beta + \left(\sum_{\beta\in\Gamma^\gamma} x_\beta\right)^2 \right) \\ &\quad - \left(\sum_{l\in\Gamma_1} x_{\alpha(l)} x_{\beta(l)} - 2\sum_{\beta\in\Gamma^\gamma} x_\gamma x_\beta + \left(\sum_{\beta\in\Gamma^\gamma} x_\beta\right)^2 \right) \\ &= \sum_{\alpha\in\Gamma_0} x_\alpha^2 - \sum_{l\in\Gamma_1} x_{\alpha(l)} x_{\beta(l)} = B(x). \end{aligned}$$

We have from Proposition 2.1 of [1] that B is positive definite, since Γ is a simply laced Dynkin diagram. It follows that the level sets of B are convex, so the set of integral points x with $B(x) = 1$ is finite. This set includes all the basis vectors of $\mathbb{Q}^{\Gamma_0}$, so there are only finitely many points each basis vector can be mapped to by an element of $\mathscr{W}$. Therefore, $\mathscr{W}$ is finite, so the set of positive roots is finite.

□

Remark 4.13. Given Proposition 4.12, to prove Theorem 4.11 all we must show is that the claimed bijection between positive roots and isomorphism classes of indecomposable representations is in fact a bijection. In particular, now it is sufficient to show that $x \in \mathbb{Q}^{\Gamma_0}$ is a positive root if and only if it is the dimension vector of some indecomposable representation.

Proving the bijection is valid will require considerable setup.

4.1. **Defining the reflection functors.** In this section, we construct for every sink $\beta \in \Gamma_0$ a functor

$$F_\beta^+ : \mathscr{L}(\Gamma, \Lambda) \to \mathscr{L}(\Gamma, \sigma_\beta\Lambda),$$

and similarly for every source α a functor

$$F_\alpha^- : \mathscr{L}(\Gamma, \Lambda) \to \mathscr{L}(\Gamma, \sigma_\alpha\Lambda).$$

Remark 4.14. The goal, roughly speaking, is for the functor F_β^+ (resp. F_α^-) to have the same effect as σ_β (resp. σ_α) on the dimension vector of most indecomposable representations it is applied to.

First, we define the functors and verify they are well-defined.

Definition 4.15. For an object $(V, f) \in \mathscr{L}(\Gamma, \Lambda)$ and a sink β, we define $F_\beta^+(V)$ to be the object $(W, g) \in \mathscr{L}(\Gamma, \sigma_\beta\Lambda)$ where

- $W_\gamma = V_\gamma$ for $\gamma \neq \beta$,
- $g_l = f_l$ for l not adjacent to β,

- $W_\beta = \ker h$, where h is the linear map
$$h : \bigoplus_{l \in \Gamma^\beta} V_{\alpha(l)} \to V_\beta$$
defined by
$$h(v_{\alpha(l_1)}, ..., v_{\alpha(l_k)}) = f_{l_1}(v_{\alpha(l_1)}) + ... + f_{l_k}(v_{\alpha(l_k)}),$$
and
- for edges l adjacent to β, the map g_l (now "pointing away" from β) is the composition of the inclusion
$$W_\beta := \ker h \to \bigoplus_{l \in \Gamma^\beta} V_{\alpha(l)}$$
with the the projection
$$\bigoplus_{l \in \Gamma^\beta} V_{\alpha(l)} \to V_{\alpha(l)}.$$

We must also define the morphism part of F_β^+. Let $\varphi : (V, f) \to (W, g)$ be a morphism of representations in the category $\mathscr{L}(\Gamma, \Lambda)$. It has components
$$\varphi_\gamma : V_\gamma \to W_\gamma$$
for each $\gamma \in \Gamma_0$. We define $F_\beta^+(\varphi)$ as a morphism
$$\psi : F_\beta^+((V, f)) \to F_\beta^+((W, g))$$
in $\mathscr{L}(\Gamma, \sigma_\beta\Lambda)$. Define
$$F_\beta^+((V, f)) = (\tilde{V}, \tilde{f})$$
and
$$F_\beta^+((W, g)) = (\tilde{W}, \tilde{g}).$$
Define h_f and h_g to be the maps used in the definitions of $\tilde{V}_\beta$ and $\tilde{W}_\beta$, respectively. Then we can define $\psi_\gamma = \varphi_\gamma$ for $\gamma \neq \beta$ and
$$\psi_\beta : \ker h_f \to \ker h_g, (v_1, ..., v_k) \mapsto (\varphi_{\alpha(l_1)}(v_1), ...\varphi_{\alpha(l_k)}(v_k)).$$
First, we verify that the image of ψ_β actually lies in $\ker h_g$. Since φ is a morphism in $\mathscr{L}(\Gamma, \Lambda)$, we have $\varphi_{\beta(l)} f_l = g_l \varphi_{\alpha(l)}$ for every edge l. It follows that for $(v_1, ..., v_k)$ in $\ker h_f$,
$$0 = \varphi_\beta(h_f(v_1, ..., v_k)) = \varphi_\beta\left(\sum_{i=1}^k f_l(v_i)\right)$$
$$= \sum_{i=1}^k \varphi_\beta f_l(v_i) = \sum_{i=1}^k g_l \varphi_{\alpha(l_i)}(v_i) = h_g(\varphi_{\alpha(l_1)}(v_1), ...\varphi_{\alpha(l_k)}(v_k)),$$
so $(\varphi_{\alpha(l_1)}(v_1), ...\varphi_{\alpha(l_k)}(v_k))$ is an element of $\ker h_g$ as desired.

It follows immediately from the definitions that ψ is a morphism of representations, and also that F_β^+ is a well-defined functor.

Definition 4.16. Similarly, for a source $\alpha \in \Gamma_0$, we define $F_\alpha^-(V)$ to be the object (W, g) of $\mathscr{L}(\Gamma, \sigma_\alpha\Lambda)$ where

- $W_\gamma = V_\gamma$ for $\gamma \neq \alpha$,
- $g_l = f_l$ for l not adjacent to α,

- $W_\alpha = \bigoplus_{l\in\Gamma^\alpha} V_{\beta(l)}/\mathrm{Im}\ \tilde{h}$, where $\tilde{h}$ is the linear map
$$\tilde{h} : V_\alpha \to \bigoplus_{l\in\Gamma^\alpha} V_{\beta(l)}$$
defined by
$$\tilde{h}(v) = (f_{l_1}(v_{\beta(l_1)}), ..., f_{l_k}(v_{\beta(l_k)})),$$
and
- for edges l adjacent to α, the map g_l (now "pointing towards" α) is the composition of the inclusion
$$V_\beta(l) \to \bigoplus_{l\in\Gamma^\alpha} V_{\beta(l)}$$
with the natural projection
$$\bigoplus_{l\in\Gamma^\alpha} V_{\beta(l)} \to \bigoplus_{l\in\Gamma^\alpha} V_{\beta(l)}/\mathrm{Im}\ \tilde{h}.$$

Again, we must define the morphism part of F_α^-. We define $F_\alpha^-(\varphi)$ to be a morphism
$$\psi : F_\alpha^-((V, f)) \to F_\alpha^-((W, g))$$
of $\mathscr{L}(\Gamma, \sigma_\alpha\Lambda)$. Define
$$F_\alpha^-((V, f)) = (\tilde{V}, \tilde{f})$$
and
$$F_\alpha^-((W, g)) = (\tilde{W}, \tilde{g}).$$
Define $\tilde{h}_f$ and $\tilde{h}_g$ to be the maps used in the definitions of $\tilde{V}_\alpha$ and $\tilde{W}_\alpha$, respectively.

Then we can define $\psi_\gamma = \varphi_\gamma$ for $\gamma \neq \alpha$, and we can define
$$\psi_\alpha : \bigoplus_{l\in\Gamma^\alpha} V_{\beta(l)}/\mathrm{Im}\ \tilde{h}_f \to \bigoplus_{l\in\Gamma^\alpha} W_{\beta(l)}/\mathrm{Im}\ \tilde{h}_g$$
by
$$\psi_\alpha([(v_1, ..., v_k)]) = [(\varphi_{\beta(l_1)}(v_1), ..., \varphi_{\beta(l_k)}(v_k))],$$
where $[v]$ is the equivalence class of a vector $[v]$ in either $\tilde{V}_\alpha$ or $\tilde{W}_\alpha$.

First, we verify that ψ_α is a well-defined function. Since φ is a morphism in $\mathscr{L}(\Gamma, \Lambda)$, we have that $\varphi_{\beta(l)} f_l = g_l \varphi_{\alpha(l)}$ for every edge l. Let $(d_1, ..., d_k) \in \mathrm{Im}\ \tilde{h}_f$. Then there exists some $v \in V_\alpha$ such that $\tilde{h}_f(v) = (f_{l_1}(v), ..., f_{l_k}(v)) = (d_1, ..., d_k)$. It follows that we have
$$\psi_\alpha([d_1, ..., d_k]) = [(\varphi_{\beta(l_1)}(d_1), ..., \varphi_{\beta(l_k)}(d_k))] = [\varphi_{\beta(l_1)}(f_{l_1}(v)), ..., \varphi_{\beta(l_k)}(f_{l_k}(v)))]$$
$$= [g_{l_1}(\varphi_\alpha(v)), ..., g_{l_k}(\varphi_\alpha(v))] = [\tilde{h}_g(\varphi_\alpha(v))] = 0,$$
so ψ_α is well-defined function.

As in the F_β^+ case, it follows immediately from the definitions that ψ is a morphism of representations and that F_α^- is a well-defined functor.

4.2. **The main theorem about the reflection functors.** In this section, we will make precise and prove the claim about the reflection functors described in Remark 4.14

Remark 4.17. Note that $F_\beta^- F_\beta^+(V)$ and $F_\alpha^+ F_\alpha^-(V)$ are well-defined, since the reflection functors turn sinks into sources and vice versa.

For any sink β of the graph Γ of a quiver (Γ, Λ), and any representation V in $\mathscr{L}(\Gamma, \Lambda)$, define a morphism in $\mathscr{L}(\Gamma, \Lambda)$

$$i_V^\beta : F_\beta^- F_\beta^+(V) \to V$$

as follows:

If $\gamma \neq \beta$, then $(F_\beta^- F_\beta^+(V))_\gamma$ is just V_γ, so we take $(i_V^\beta)\gamma = \mathrm{Id}$.

For the definition of $(i_V^\beta)_\beta$ we use the fact that in the sequence

$$F_\beta^+(V)_\beta \xrightarrow{\tilde{h}} \bigoplus_{l \in \Gamma^\beta} V_{\alpha(l)} \xrightarrow{h} V_\beta,$$

we have $\ker h = \mathrm{Im}\ \tilde{h}$.

Remark 4.18. This fact is worth clarifying. The map h is the one involved in defining $F_\beta^+(V)$, and the map $\tilde{h}$ is the one involved in defining $F_\beta^- F_\beta^+(V)$. The space $F_\beta^+(V)_\beta$ is equal to $\ker h$ by definition, which is a subspace of $\bigoplus_{l \in \Gamma^\beta} V_{\alpha(l)}$. Since the maps out of the space at β in the representation $F_\beta^+(V)$ are essentially projections [1] onto the spaces $V_{\alpha(l)}$, here $\tilde{h}$ is just the natural inclusion map. Thus, the observation that $\ker h = \mathrm{Im}\ \tilde{h}$ is just the observation that mapping $\ker h$ into the domain of h by an inclusion yields $\ker h$.

Thus, we can define the component map

$$(i_V^\beta)_\beta : F_\beta^- F_\beta^+(V)_\beta = \bigoplus_{l \in \Gamma^\beta} V_{\alpha(l)}/\mathrm{Im}\tilde{h} = \bigoplus_{l \in \Gamma^\beta} V_{\alpha(l)}/\ker h \to V_\beta$$

in a natural way:

$$[v_1, ..., v_k] \mapsto h(v_1, ..., v_k).$$

It is easy to verify that i_V^β is a morphism.

Similarly, for any source α, define another morphism in $\mathscr{L}(\Gamma, \Lambda)$, denoted

$$p_V^\alpha : V \to F_\alpha^+ F_\alpha^-(V),$$

as follows: For $\gamma \neq \alpha$, as before we can take $(p_V^\alpha)_\gamma = \mathrm{Id}$. To define $(p_V^\alpha)_\alpha$, we can use the fact that

$$F_\alpha^+ F_\alpha^-(V)_\alpha = \ker h,$$

where h is the projection

$$h : \bigoplus_{l \in \Gamma^\alpha} V_{\beta(l)} \to \bigoplus_{l \in \Gamma^\alpha} V_{\beta(l)}/\mathrm{Im}\ \tilde{h},$$

since the maps pointing inward to α in $F_\alpha^-(V)$ whose sum defines h are themselves projections onto $\bigoplus_{\beta \in \Gamma^\alpha} V_\beta/\mathrm{Im}\ \tilde{h}$. Thus

$$F_\alpha^+ F_\alpha^-(V)_\alpha = \mathrm{Im}\ \tilde{h},$$

[1] Really, these maps are compositions of projections and inclusions. Throughout the rest of the section, we often refer to such maps as simply "projections" to avoid awkward phrasings.

so we can simply define $(p_V^\alpha)_\alpha = \tilde{h}$.

It is similarly easy to verify that p_V^α is a morphism. Next, we show some useful properties of i_V^β and p_V^α.

Lemma 4.19. *(1) p_V^α is an epimorphism and i_V^β is a monomorphism.*

(2) If V is of the form $F_\alpha^+(W)$ for some W, then p_V^α is an isomorphism, and similarly, if V is of the form $F_\beta^-(W)$, then i_V^β is an isomorphism.

Proof. (1) First, we show that i_V^β is monic. Let ϕ, ψ be parallel morphisms from some object $(W, g) \in \mathscr{L}(\Gamma, \Lambda)$ to $F_\beta^- F_\beta^+(V)$. We need to show

$$i_V^\beta \phi = i_V^\beta \psi \implies \phi = \psi.$$

This follows from the fact that every component of i_V^β is an injection.

Similarly, p_V^α is epic, since every component of p_V^α is a surjection.

(2) Whenever two reflection functors of opposite sign and of the same vertex are applied (for example $F_\alpha^- F_\alpha^+$), we have $\ker h = \mathrm{Im}\ \tilde{h}$. Therefore, when we apply a sequence of three reflection functors of α to some representation, the spaces at α in the representations we obtain after the first and third iterations are equal. For example, the image of the map $\tilde{h}$ in the definition of the space $(F_\alpha^+ F_\alpha^- F_\alpha^+(W))_\alpha$ is the same as the image of the map $\tilde{h}$ in the definition of the space $(F_\alpha^+(W))_\alpha$, so both spaces can be written as $\ker h$ or $\mathrm{Im}\ \tilde{h}$ without confusion. Now, the component map

$$(p_V^\alpha)_\alpha : \ker h \to \mathrm{Im}\ \tilde{h}$$

of the morphism

$$p_V^\alpha : F_\alpha^+(W) \to F_\alpha^+ F_\alpha^- F_\alpha^+(W)$$

is defined as a restriction of the map $\tilde{h}$ from the definition of $(F_\alpha^-(F_\alpha^+(W)))_\alpha$. But this $\tilde{h}$ is just a direct sum of the projections on edges adjacent to α that appear in $F_\alpha^+(W)$, so $(p_V^\alpha)_\alpha$ is an inclusion map. Further, since $\ker h = \mathrm{Im}\ \tilde{h}$, we know that the domain of $(p_V^\alpha)_\alpha$ is all of $\mathrm{Im}\ \tilde{h}$, so $(p_V^\alpha)_\alpha$ is the identity and thus an isomorphism.

Similarly, the component map

$$(i_V^\beta)_\beta : \bigoplus_{\alpha \in \Gamma^\beta} V_\alpha / \ker h \to \bigoplus_{\alpha \in \Gamma^\beta} V_\alpha / \mathrm{Im}\ \tilde{h}$$

of the morphism

$$i_V^\beta : F_\beta^- F_\beta^+ F_\beta^-(W) \to F_\beta^-(W)$$

is defined as a restriction of the map h from the definition of $(F_\beta^+(F_\beta^-(W)))_\beta$. But this h is a sum of the projections on edges adjacent to β that appear in $F_\beta^-(W)$, so it is an inclusion. Further, since $\ker h = \mathrm{Im}\ \tilde{h}$, we similarly obtain that $(i_V^\beta)_\beta$ is the identity.

□

Now we can state and prove the main theorem about the reflection functors.

Theorem 4.20. *Let (Γ, Λ) be a quiver, and let $V \in \mathscr{L}(\Gamma, \Lambda)$ be an indecomposable object. Then*

(1) For any sink $\beta \in \Gamma_0$ with respect to Λ, one of the following holds:

- *$V \cong L_\beta$ and $F^+_\beta V = 0$.*
- *$F^+_\beta(V)$ is an indecomposable object, $F^-_\beta F^+_\beta(V) \cong V$, and the dimension vector of $F^+_\beta(V)$ can be calculated by applying σ_β to the dimension vector of V.*

(2) Similarly, for any source $\alpha \in \Gamma_0$ with respect to Λ, one of the following holds:
- *$V \cong L_\alpha$ and $F^+_\alpha V = 0$.*
- *$F^+_\alpha(V)$ is an indecomposable object, $F^+_\alpha F^-_\alpha(V) \cong V$, and the dimension vector of $F^-_\alpha(V)$ can be calculated by applying σ_α to the dimension vector of V.*

Remark 4.21. A rougher but more intuitive phrasing of this theorem is the following: Applying the reflection functor of a sink or source γ to an indecomposable representation V yields another indecomposable representation with dimension and orientation modified by σ_γ, except when V is the simple representation L_γ. Moreover, in the former case, the two reflection functors of γ are inverses, and in the latter case, the reflection functor sends L_γ to 0.

Proof. To begin the proof of (1), for any sink β we can write

$$V \cong F^-_\beta F^+_\beta(V) \oplus V/\mathrm{Im}\ i^\beta_V.$$

Since V is indecomposable, then it must coincide with one of the summands.

- Case 1: Suppose we have $V = V/\mathrm{Im}\ i^\beta_V$. Here, since the components of i^β_V at vertices other than β are identity maps, we have that the only nonzero space of $V/\mathrm{Im}\ i^\beta_V$ is the one at β. Since V is indecomposable, we must have $V = L_\beta$. Therefore, $F^+_\beta(V) = 0$.
- Case 2: Suppose we have $V = F^-_\beta F^+_\beta(V)$. Here, since $V/\mathrm{Im}\ i^\beta_V = 0$ and all the components of i^β_V are injective, we have that i^β_V is an isomorphism. In particular, we have

$$\dim V_\beta = \dim\left(\bigoplus_{l\in\Gamma^\beta} V_{\alpha(l)}/\ker h\right).$$

Since $F^+_\beta(V)_\beta = \ker h$, it follows that we have

$$\dim F^+_\beta(V)_\beta = -\dim V_\beta + \sum_{l\in\Gamma^\beta} \dim V_{\alpha(l)}.$$

Moreover, the dimensions of the other component spaces of $F^+_\beta(V)$ and V clearly agree, so we have

$$\dim F^+_\beta(V) = \sigma_\beta \dim V.$$

It remains to show that $F^+_\beta(V)$ is indecomposable. Suppose $F^+_\beta(V) = W_1 \oplus W_2$. Then we have

$$V = F^-_\beta(W_1 \oplus W_2) = F^-_\beta(W_1) \oplus F^-_\beta(W_2),$$

so one of the summands, say $F^-_\beta(W_2)$, must be 0. We have from Lemma 4.19 that the morphism

$$p^\beta_V : F^+_\beta(V) \to F^+_\beta F^-_\beta(F^+_\beta(V))$$

is an isomorphism. Thus we have

$$p_V^{\beta}(W_2) \subset F_\beta^+ F_\beta^-(W_2) = 0,$$

so $W_2 = 0$ and $F_\beta^+(V)$ is indecomposable.

A similar argument, using the decomposition $V \cong F_\alpha^+ F_\alpha^-(V) \oplus \ker p_V^\alpha$, suffices to show part (2) of the theorem.

□

4.3. **Some corollaries of Theorem 4.20.** Let (Γ, Λ) be an oriented graph and let $\alpha_1, ..., \alpha_k$ be a sink sequence.

Corollary 4.22. *For any $i \in [k]$, the object $F_{\alpha_1}^- ... F_{\alpha_{i-1}}^-(L_{\alpha_i})$ is either 0 or an indecomposable object in $\mathscr{L}(\Gamma, \Lambda)$.*

Remark 4.23. Note that here L_{α_i} is considered as an object of $\mathscr{L}(\Gamma, \sigma_{\alpha_{i-1}} ... \sigma_{\alpha_1} \Lambda)$ rather than $\mathscr{L}(\Gamma, \Lambda)$. In general, the simple representations should be considered as objects of the appropriate categories for applications of the reflection functors to make sense.

Corollary 4.24. *Let $V \in \mathscr{L}(\Gamma, \Lambda)$ be an indecomposable object such that we have*

$$F_{\alpha_k}^+ F_{\alpha_{k-1}}^+ ... F_{\alpha_1}^+(V) = 0.$$

Then for some i, we have

$$V \cong F_{\alpha_1}^- F_{\alpha_2}^- ... F_{\alpha_{i-1}}^-(L_{\alpha_i}),$$

since we have both

$$F_{\alpha_{i-1}}^+ ... F_{\alpha_1}^+ F_{\alpha_1}^- ... F_{\alpha_{i-1}}^-(L_{\alpha_i}) \cong L_{\alpha_i}$$

and that the only indecomposable object mapped to 0 by any $F_{\alpha_i}^+$ is L_{α_i}. Indeed, L_{α_i} is the only indecomposable object mapped to any decomposable object by any $F_{\alpha_i}^+$.

Corollary 4.25. *Let V be an indecomposable object. Let*

$$V_j = F_{\alpha_j}^+ ... F_{\alpha_1}^+ V, \quad m_j = \sigma_{\alpha_j} ... \sigma_{\alpha_1}(\dim V)$$

for $0 \leq j \leq k$. Let i be the last index such that m_j is positive for all $j \leq i$. Then the V_j are indecomposable objects for $j \leq i$, and $V = F_{\alpha_1}^- ... F_{\alpha_i}^-(L_{\alpha_{i+1}})$.

Statements similar to each of the above are true when $+$ is replaced by $-$ and the α_i form a source sequence.

4.4. **Coxeter transformations.**

Definition 4.26. Let $\alpha_1, ..., \alpha_n$ be an ordering of all the vertices of a graph Γ. The linear transformation

$$c := \sigma_{\alpha_n} ... \sigma_{\alpha_1} : \mathbb{Q}^{\Gamma_0} \to \mathbb{Q}^{\Gamma_0}$$

is called a **Coxeter transformation**.

We now prove several useful facts about Coxeter transformations under the assumption that Γ is a simply laced Dynkin diagram, and thus that B is positive definite.

Proposition 4.27. *The Coxeter transformation c has no nonzero fixed points in $\mathbb{Q}^{\Gamma_0}$.*

Proof. We use the symmetric bilinear form $\langle , \rangle$ associated with the quadratic form B, which is defined using the polarization identity

$$\langle x, y \rangle = \frac{1}{2}(B(x+y) - B(x) - B(y)).$$

From this we can deduce that if α, β are distinct vertices, then $\langle \overline{\alpha}, \overline{\alpha} \rangle = 1$ and $2\langle \overline{\alpha}, \overline{\beta} \rangle$ is the negation of the number of edges joining α and β. Therefore, for any vertex β, we have

$$\sigma_\beta(x) = x - (x_\beta - \sum_{l \in \Gamma^\beta} x_{\gamma(l)})\overline{\beta} = x - 2\langle x, \overline{\beta} \rangle \overline{\beta}.$$

Now suppose x is a nonzero fixed point of c. Since σ_{α_1} is the only transformation among the σ_{α_i}'s that can change the α_1 component of x, we must have $(\sigma_{\alpha_1} x)_{\alpha_1} = x_{\alpha_1}$, and thus $\sigma_{\alpha_1} x = x$. Similarly we can show $\sigma_{\alpha_i} x = x$ for all $1 \leq i \leq n$. Therefore for all i, we have

$$x - \langle x, \overline{\alpha_i} \rangle \overline{\alpha_i} = x,$$

and so, we obtain $\langle x, \overline{\alpha_i} \rangle = 0$ for all i. But this is a contradiction since B is positive definite and the $\overline{\alpha_i}$ form a basis for $\mathbb{Q}^{\Gamma_0}$. □

Proposition 4.28. *If $x \in \mathbb{Q}^{\Gamma_0}$ is nonzero, then for some i the vector $c^i x$ is not positive.*

Proof. Since the group $\mathscr{W}$ is finite, there exists h such that $c^h = 1$. If all the $c^i x$ are positive for some nonzero x, then $y = x + cx + ... + c^{h-1}x$ is a nonzero invariant of c, contradicting Proposition 4.27. □

4.5. **Returning to the bijection and the proof of Theorem 4.11.** Now we are ready to finish the proof of Gabriel's Theorem.

Theorem 4.29. *The dimension vector of every indecomposable representation V is a positive root.*

Proof. Let $\alpha_1, ..., \alpha_n$ be a sink ordering of Γ_0, and let $c = \sigma_{\alpha_n} ... \sigma_{\alpha_1}$ be the corresponding Coxeter transformation. We have from Proposition 4.28 that for some k, the vector $c^k(\dim V)$ is nonpositive. Considering the sink sequence $\beta_1, ..., \beta_{nk}$ defined to be $(\alpha_1, ..., \alpha_n)$ repeated k times, we obtain that $\sigma_{\beta_{nk}} ... \sigma_{\beta_1}(x) = c^k(x)$ is nonpositive. Therefore, by Theorem 4.20, the representation $F^+_{\beta_{nk}} ... F^+_{\beta_1}(V)$ is 0.

Thus, from Corollary 4.24, it follows that there is an index $i < kn$ such that we have

$$V = F^-_{\beta_1} ... F^-_{\beta_i}(L_{\beta_{i+1}})$$

and thus we also have that the dimension vector of V is $\sigma_{\beta_1} ... \sigma_{\beta_i}(\overline{\beta}_{i+1})$, a positive root.

□

Lemma 4.30. *The only root that σ_β maps to a nonpositive vector is $\overline{\beta}$.*

Proof. Since B and the integral lattice are both preserved by the σ_β, all roots x are integral and have $B(x) = 1$. Recall from the proof of Proposition 4.27 that we have $\sigma_\beta(x) = x - 2\langle x, \overline{\beta} \rangle \overline{\beta}$.

Defining $\|x\| = \sqrt{B(x)}$, for any root x we have $\|x\| = \|\overline{\beta}\| = 1$. So, from the Cauchy-Schwarz inequality, we obtain $|\langle x, \overline{\beta} \rangle| \leq 1$. Moreover, $2\langle x, \overline{\beta} \rangle \in \mathbb{Z}$ since both vectors are integral. Thus $\langle x, \overline{\beta} \rangle$ is 2, 1, 0, -1, or -2.

- If $2\langle x, \overline{\beta}\rangle = 2$, then $x = \overline{\beta}$.
- If $2\langle x, \overline{\beta}\rangle = 1$, then since $2\langle x, \overline{\beta}\rangle = x_\beta - \sum_{l \in \Gamma^\beta} x_{\gamma(l)}$, it must be that $x_\beta > 0$, and x and $\overline{\beta}$ must differ somewhere, so $\sigma_\alpha x = x - \langle x, \overline{\beta}\rangle\overline{\beta} = x - \overline{\beta}$ is positive.
- If $2\langle x, \overline{\beta}\rangle \leq 0$, then $\sigma_\beta x = x - 2\langle x, \overline{\beta}\rangle\overline{\beta} > 0$.

□

Remark 4.31. We now have a good test for whether a given representation of a quiver whose underlying graph is a simply laced Dynkin diagram is indecomposable – simply plug its dimension vector into B and see whether you get 1! We know that if x is a root, then $B(x) = 1$, and to verify the converse we only need to check finitely many cases.

Theorem 4.32. *Every positive root $x \in \mathbb{Q}^{\Gamma_0}$ is the dimension vector of some indecomposable object of $\mathscr{L}(\Gamma, \Lambda)$.*

Remark 4.33. Note that if x is the dimension vector of some indecomposable object, then it is the dimension vector of the entire isomorphism class of that indecomposable object.

Proof. As in Theorem 4.29, let $\alpha_1, \ldots, \alpha_n$ be a sink ordering of Γ_0, and let $c = \sigma_{\alpha_1} \ldots \sigma_{\alpha_n}$ be the corresponding Coxeter transformation. By Proposition 4.28, there exists k such that $c^k x$ is nonpositive. Taking (β_i) to be the sink sequence consisting of $(\alpha_1, \ldots, \alpha_n)$ repeated k times, we obtain that $\sigma_{\beta_{nk}} ... \sigma_{\beta_1}(x) := c^k x$ is nonpositive.

Let i be the last index for which $\sigma_{\beta_i} ... \sigma_{\beta_1}(x) > 0$. Then we have from Lemma 4.30 that

$$\sigma_{\beta_i} ... \sigma_{\beta_1}(x) = \overline{\beta_{i+1}}.$$

It follows from Corollary 4.25 and Theorem 4.20 that $V := F^-_{\beta_i} ... F^-_{\beta_1}(L_{\beta_{i+1}})$ is an indecomposable object with dimension vector $\sigma_{\beta_1} ... \sigma_{\beta_i}(\overline{\beta_{i+1}}) = x$.

□

Theorem 4.29 and Theorem 4.32 together are equivalent to Theorem 4.11, so this concludes the proof of Gabriel's Theorem.

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References

[1] I.N. Bernstein, I.M. Gelfand, V.A. Ponomarev. Coxeter Functors and Gabriel's Theorem, 1972.
[2] Alexander Kirillov, Jr. Quiver Representations and Quiver Varieties. American Mathematical Society, 2016.
[3] T.Y. Lam. A First Course in Noncommutative Rings, Springer, 1991.
[4] T.Y. Lam. Lectures on Modules and Rings, Springer, 1999.